Enriched structure–semantics adjunctions and monad–theory equivalences for subcategories of arities

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Motivation

- Several structure–semantics adjunctions and monad–theory equivalences have been established in category theory.
- In [6, 7], Lawvere and Linton established a structure–semantics adjunction between Lawvere theories and *tractable* **Set**-valued functors. For a complete symmetric monoidal closed category $\mathscr{V}$, Dubuc [4] established a structure–semantics adjunction between $\mathscr{V}$*-theories* and *tractable* $\mathscr{V}$-valued $\mathscr{V}$-functors.

Motivation

- Linton [7] also showed that there is an equivalence between Lawvere theories and finitary monads on **Set**. Lucyshyn-Wright [8] generalized this result by showing that if $\mathscr{J} \hookrightarrow \mathscr{V}$ is any *eleutheric system* of arities in a closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$*-theories* and $\mathscr{J}$*-ary* $\mathscr{V}$-monads on $\mathscr{V}$.

- Building on work of Power and Nishizawa [11], Bourke and Garner [2] recently showed that if $\mathscr{J} \hookrightarrow \mathscr{C}$ is any small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ enriched over a locally presentable closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$*-theories* and $\mathscr{J}$*-nervous* $\mathscr{V}$-monads on $\mathscr{C}$.

Objectives

- Neither monad–theory equivalence of Lucyshyn-Wright [8] or Bourke–Garner [2] subsumes the other; can both equivalences, along with the aforementioned structure–semantics adjunctions, be captured and unified by a common framework that also provides new examples?

- Yes! Given a subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ enriched over a closed category $\mathscr{V}$, we will identify hypotheses on these data that entail a structure–semantics adjunction, a monad–theory equivalence, and a rich theory of *presentations* for enriched monads and theories.

- Moral of the story: these "nice" subcategories of arities admit extremely rich and useful treatments of enriched algebra, in some completely new settings.

Basic definitions

- We fix a **subcategory of arities** $j : \mathscr{J} \hookrightarrow \mathscr{C}$, i.e. a small, full, and dense sub-$\mathscr{V}$-category, in a $\mathscr{V}$-category $\mathscr{C}$ enriched over a complete and cocomplete symmetric monoidal closed category $\mathscr{V}$.

- We have a fully faithful $\mathscr{V}$-functor

$$N_j : \mathscr{C} \to [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]$$

$$N_j C = \mathscr{C}(j-, C)$$

that we call the j-**nerve** $\mathscr{V}$-functor. The presheaves in its essential image are called j-**nerves**.

Pretheories and their algebras

- (Linton [7], Diers [3], Bourke–Garner [2]) A $\mathscr{J}$**-pretheory** is an identity-on-objects $\mathscr{V}$-functor $\tau : \mathscr{J}^{\mathbf{op}} \to \mathscr{T}$, while a $\mathscr{J}$**-theory** is a $\mathscr{J}$-pretheory $\mathscr{T}$ such that each $\mathscr{T}(J, \tau-) : \mathscr{J}^{\mathbf{op}} \to \mathscr{V}$ ($J \in \mathbf{ob}\,\mathscr{J}$) is a j-nerve. We have the category $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-pretheories and its full subcategory $\mathbf{Th}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-theories.

- Let $\mathscr{T}$ be a $\mathscr{J}$-pretheory. The $\mathscr{V}$-category $\mathscr{T}$-Alg of $\mathscr{T}$**-algebras** is defined by the following pullback in $\mathscr{V}$-CAT:

$$\begin{array}{ccc} \mathscr{T}\text{-Alg} & \longrightarrow & [\mathscr{T}, \mathscr{V}] \\ {\scriptstyle U^{\mathscr{T}}}\downarrow & & \downarrow{\scriptstyle [\tau,1]} \\ \mathscr{C} & \xrightarrow[N_j]{} & [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]. \end{array}$$

Amenable subcategories of arities

- A $\mathscr{J}$-pretheory $\mathscr{T}$ is **admissible** if $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}$ has a left adjoint (i.e. if $\mathscr{T}$ **admits** free algebras).
- The subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ is **amenable** if every $\mathscr{J}$-theory is admissible, and is **strongly amenable** if every $\mathscr{J}$-pretheory $\mathscr{T}$ is admissible.

$\mathscr{J}$-tractable $\mathscr{V}$-categories

- An object $G : \mathscr{A} \to \mathscr{C}$ of $\mathscr{V}$-CAT$/\mathscr{C}$ is a **$\mathscr{J}$-tractable $\mathscr{V}$-category over** $\mathscr{C}$ if $\mathscr{C}$ admits the weighted limit $\{\mathscr{C}(J, G-), G\}$ for each $J \in \mathbf{ob}\,\mathscr{J}$. Then $\mathscr{J}$-$\mathbf{Tract}(\mathscr{C})$ is the full subcategory of $\mathscr{V}$-CAT$/\mathscr{C}$ consisting of the $\mathscr{J}$-tractable $\mathscr{V}$-categories over $\mathscr{C}$.

- Let $\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ be the full subcategory of $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ consisting of the *admissible* $\mathscr{J}$-pretheories. We can then define a **semantics** functor

$$\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$$

by

$$\mathbf{Sem}\,\mathscr{T} = \left(U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}\right).$$

$\mathscr{J}$-structure

- Let $G : \mathscr{A} \to \mathscr{C}$ be a $\mathscr{J}$-tractable $\mathscr{V}$-category over $\mathscr{C}$. We define a $\mathscr{J}$-theory $\tau_G : \mathscr{J}^{\mathbf{op}} \to \mathbf{Str}\,G$, the **$\mathscr{J}$-structure of G**, by taking the (identity-on-objects, fully faithful) factorization of the composite $\mathscr{V}$-functor

$$\begin{array}{ccccc} \mathscr{J}^{\mathbf{op}} & \xrightarrow{\;j^{\mathbf{op}}\;} & \mathscr{C}^{\mathbf{op}} & \xrightarrow{\;N_{G^{\mathbf{op}}}\;} & [\mathscr{A}, \mathscr{V}]. \\ & \searrow^{\tau_G} & & \nearrow & \\ & & \mathbf{Str}\,G & & \end{array}$$

The structure–semantics adjunction

Theorem

Let $\mathscr{J} \hookrightarrow \mathscr{C}$ *be an amenable subcategory of arities. Then the semantics functor* $\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$ *has a left adjoint* **Str** *that sends each* $\mathscr{J}$*-tractable* $\mathscr{V}$*-category over* $\mathscr{C}$ *to its* $\mathscr{J}$*-structure. This adjunction is idempotent, and also induces an idempotent* **monad–pretheory** *adjunction*

$$\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Mnd}(\mathscr{C}),$$

where Φ *sends a* $\mathscr{V}$*-monad* $\mathbb{T}$ *to its* **Kleisli** $\mathscr{J}$**-theory**, *while* Ψ *sends an admissible* $\mathscr{J}$*-pretheory* $\mathscr{T}$ *to its free* $\mathscr{T}$*-algebra* $\mathscr{V}$*-monad.*

The monad–theory equivalence

A $\mathscr{V}$-monad $\mathbb{T}$ on $\mathscr{C}$ is $\mathscr{J}$**-nervous** if $\mathbb{T} \cong \Psi\mathscr{T}$ for some admissible $\mathscr{J}$-pretheory $\mathscr{T}$ (there are other equivalent definitions too).

Theorem

Let $\mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. Then the idempotent monad–pretheory adjunction $\Psi \dashv \Phi$ restricts to an adjoint equivalence

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$$

between $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads, which commutes with semantics in an appropriate sense.

Additional consequences of strong amenability

Theorem

Let $\mathscr{C}$ be complete and cocomplete, and suppose that $\mathscr{J} \hookrightarrow \mathscr{C}$ is strongly amenable. Then:

1. $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$, $\mathbf{Th}_{\mathscr{J}}(\mathscr{C})$, *and* $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *are all* **algebraically** *cocomplete.*
2. $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *is monadic over a category of* $\mathscr{J}$**-signatures**, *so that* $\mathscr{J}$*-nervous* $\mathscr{V}$*-monads admit a rich and useful theory of* **presentations**.

First example: bounded and eleutheric subcategories of arities

- A subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ is **eleutheric** [8, 10] if $\mathscr{C}$ is a free cocompletion of $\mathscr{J}$ under a class of weighted colimits, and it is **bounded** [10] if each $J \in \mathbf{ob}\,\mathscr{J}$ is suitably "compact" or "presentable". For example:
  - The full sub-$\mathscr{V}$-category of enriched α-presentable objects in a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ enriched over a locally α-presentable $\mathscr{V}$.
  - The "strongly finitary" subcategory of arities $\mathbf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the terminal object (i.e. the natural number arities) in any complete and cocomplete cartesian closed category $\mathscr{V}$.
  - The Yoneda embedding $\mathbf{y} : \mathscr{A}^{\mathbf{op}} \hookrightarrow [\mathscr{A}, \mathscr{V}]$ for a small $\mathscr{V}$-category $\mathscr{A}$.

First example: bounded and eleutheric subcategories of arities

Theorem

Every eleutheric subcategory of arities is amenable, and every bounded and eleutheric subcategory of arities is strongly amenable.

For suitable choices of eleutheric (and bounded) $\mathscr{J} \hookrightarrow \mathscr{C}$, we then recover the enriched structure–semantics adjunctions and monad–theory equivalences of Lawvere and Linton [6, 7], Dubuc [4], Lucyshyn-Wright [8], and Bourke–Garner [2].

By omitting eleuthericity and strengthening the notion of boundedness, we can also obtain other classes of examples of strongly amenable subcategories of arities.

New example in locally bounded closed categories

A $\mathscr{V}$-category $\mathscr{C}$ is $\mathscr{V}$-**sketchable** if it is equivalent to the $\mathscr{V}$-category of models of a small weighted limit theory.

Theorem

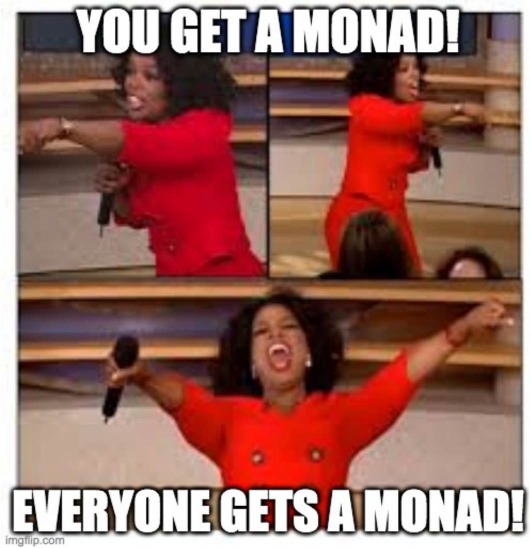
Every subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-sketchable $\mathscr{V}$-category $\mathscr{C}$ enriched over a locally bounded closed category $\mathscr{V}$ is strongly amenable. In particular, every subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{V}$ in a locally bounded closed category $\mathscr{V}$ is strongly amenable.

New example in locally bounded closed categories

We have the following examples (and more!) of locally bounded closed categories [9]:

- Every locally presentable closed category (incl. every Grothendieck topos, **Set**, **Cat**, **Pos**, **Ab**, **Met**...).
- Every cocomplete locally cartesian closed category with a small generator (e.g. Dubuc's categories of concrete sheaves [5] and the convenient categories of smooth spaces of Baez–Hoffnung [1], incl. simplicial complexes).
- Every symmetric monoidal closed topological category over **Set**.
- Many convenient (cartesian closed) categories of topological spaces.

All of these closed categories will now admit extremely rich and useful treatments of enriched algebra. For example, we can now construct enriched monads and theories almost "at will" on many closed categories of relevance for topology, differential geometry, analysis, and programming language semantics, many of which we have seen at ACT (**sSet**, **Poly**, **Qbs**, **DCPO**...).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In conclusion

- We have developed a general axiomatic framework for enriched structure–semantics adjunctions and monad–theory equivalences for subcategories of arities, which subsumes most (all?) known results of this kind and provides some completely new classes of examples.

- As seen in Rory's talk, we also have extremely flexible and practical methods for easily constructing enriched monads and theories in these classes of examples, using enriched operations and equations.

- If you have a favourite symmetric monoidal closed category $\mathscr{V}$ (or $\mathscr{V}$-category) on which you want to construct and study enriched monads, please talk to us! :)

Thank you!

Comments and questions are welcome!

References I

[1] John C. Baez and Alexander E. Hoffnung, *Convenient categories of smooth spaces*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5789–5825.

[2] John Bourke and Richard Garner, *Monads and theories*, Adv. Math. **351** (2019), 1024–1071.

[3] Yves Diers, *Foncteur pleinement fidéle dense classant les algébres*, Cahiers Topologie Géom. Différentielle Catég. **17** (1976), no. 2, 171–186.

[4] Eduardo J. Dubuc, *Enriched semantics-structure (meta) adjointness*, Rev. Un. Mat. Argentina **25** (1970/71), 5–26.

[5] ________, *Concrete quasitopoi*, Applications of sheaves, Lecture Notes in Math., vol. 753, Springer, Berlin, 1979, pp. 239–254.

References II

[6] F. William Lawvere, *Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories*, Repr. Theory Appl. Categ. (2004), no. 5, 1–121.

[7] F. E. J. Linton, *An outline of functorial semantics*, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 7–52.

[8] Rory B. B. Lucyshyn-Wright, *Enriched algebraic theories and monads for a system of arities*, Theory Appl. Categ. **31** (2016), No. 5, 101–137.

[9] Rory B. B. Lucyshyn-Wright and Jason Parker, *Locally bounded enriched categories*, Theory Appl. Categ. **38** (2022), No. 18, 684–736.

[10] Rory B.B. Lucyshyn-Wright and Jason Parker, *Presentations and algebraic colimits of enriched monads for a subcategory of arities*, Preprint, arXiv:2201.03466, 2022.

References III

[11] Koki Nishizawa and John Power, *Lawvere theories enriched over a general base*, J. Pure Appl. Algebra **213** (2009), no. 3, 377–386.